De Rham Cohomology, Degree, Mayer-Vietoris exact sequence

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Exercise 1. Let $A \in \mathcal{M}_n(\mathbb{Z})$. Let $f_A : \mathbb{T}^n \to \mathbb{T}^n$ be the map induced by A on the n-dimensional torus $\mathbb{T}^n$. Compute the degree of f_A.

Exercise 2. Let M, N, P be three connected and oriented manifolds of the same degree. Let $f \in \mathcal{C}^\infty(M, N)$ and $g \in \mathcal{C}^\infty(N, P)$ be two proper maps. Prove that $\deg(g \circ f) = \deg(g)\deg(f)$.

Exercise 3 (About the degree on $\mathbb{S}^2$)**.**

1. Show that every complex polynomial of degree n gives rise to a map of the sphere $\mathbb{S}^2$ to itself of degree n.
   Hint : prove that $(t, z) \mapsto a_n z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$ *gives rise to an homotopy between the map correponding to* P *and the map corresponding to* $a_n z^n$ *on the sphere* $\mathbb{S}^2$.
2. Deduce the fundamental theorem of algebra.
3. Deduce that there are maps of every degree on $\mathbb{S}^2$.

Exercise 4. Let M be a compact manifold. Compare the de Rham Cohomology of M and that of $M \setminus \{p\}$.

Exercise 5 (Hopf invariant)**.** Let $n > 1$ We consider the oriented spheres $\mathbb{S}^n$ and $\mathbb{S}^{2n-1}$. Let $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ be a smooth map.

1. Let ω be a volume form on $\mathbb{S}^n$ such that $\int_{\mathbb{S}^n} \omega = 1$. Show that there exists a $(n-1)$-form β on $\mathbb{S}^{2n-1}$ such that $f^*\omega = d\beta$.
2. Show that $\int_{\mathbb{S}^{2n-1}} \beta \wedge \mathrm{d}\beta$ doesn't depend on the choice of β. Let's denote it by $H(f, \omega)$.
3. Show that if ω' is another volume form on $\mathbb{S}^n$ such that $\int_{\mathbb{S}^n} \omega = 1$, then $H(f, \omega) = H(f, \omega')$. Let's denote is by $H(f)$. It is called the *Hopf invariant* of f.
4. Show that if $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ and $g : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ are homotopic, then $H(f) = H(g)$.
5. Compute $H(f)$ when f is a constant map.
6. Let $f : \mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z| + |w| = 1\} \to \mathbb{S}^2 = \hat{\mathbb{C}}$ be defined by $f(z, w) = z/w$ if $w \neq 0$ and $f(z, 0) = \infty$. Show that $H(f) = 1$.

Exercise 6. Let $\mathbb{RP}^n$ be the n-dimensional real projective space. Recall that $\mathbb{RP}^n = \mathbb{S}^n/\{Id, \sigma\}$, where $\sigma : x \to -x$ is the antipodal map.

1. Prove that for all $p \geqslant 0$, $\Omega^p(\mathbb{RP}^n)$ is isomorphic to $\{\omega \in \Omega^p(\mathbb{S}^n) \mid \sigma^*\omega = \omega\}$.
2. Deduce the de Rham cohomology of $\mathbb{RP}^n$.

Exercise 7 (Jordan theorem)**.** Let $M \subset \mathbb{R}^n$ be a closed connected submanifold of dimension $n - 1$. For all $y \in \mathbb{R}^n \setminus M$, let $f_y : M \to \mathbb{S}^{n-1}$ be the map

$$f_y(x) := \frac{x - y}{\|x - y\|}, \quad \forall x \in M.$$

1. Show that f_y is a smooth map and compute its differential at a point $x \in M$.
2. Show that $v \in \mathbb{S}^{n-1}$ is a regular value of f_y if and only if the ray $y + \mathbb{R}_+ v$ intersects M transversally.
3. Find two points $y, z \in \mathbb{R}^n \setminus M$ close to M such that $|\deg f_z - \deg f_y| = 1$.
4. Using connectivity of M, show that $\mathbb{R}^n \setminus M$ has at most to connected component.
5. Deduce that $\mathbb{R}^n \setminus M$ has two connected component and describe them using degree.

Exercise 8 (Moser's trick)**.** Let $f : M \to N$ be a diffeomorphism and α and β be volume forms of M and N respectively. We assume that M and N have dimension n and

$$\int_M \alpha = \int_N \beta.$$

We want to prove that there exists a diffeomorphism $g : M \to N$ isotopic to f such that $g^*\beta = \alpha$.

1. For $t \in [0, 1]$ let $\nu_t := (1 - t)\alpha + tf^*\beta$. Show that (ν_t) is a smooth family of volume forms on M and that there exists $\gamma \in \Omega^{n-1}(M)$ such that $\dot{\nu}_t = \mathrm{d}\gamma$ for all $t \in [0, 1]$.
2. Find a non-autonomous vector field (X_t) on M such that its induced flow (φ_t) satisfies $\varphi_t^*\nu_t = \alpha$ for all $t \in [0, 1]$.
3. Conclude.